

Long waves on a thin layer of conducting fluid flowing down an inclined plane in an electromagnetic field

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ABSTRACT. – Wave formation on film flow is an intriguing hydrodynamic phenomenon with a variety of practical consequences, especially in heat and mass transfer. In this paper the propagation of weakly nonlinear waves over a flow of an electrically conducting viscous film flowing down an inclined plane under simultaneous action of electrical and magnetic fields is studied. The set of Navier-Stokes equations with electromagnetic force in the limit of low magnetic Reynolds number and subject to corresponding boundary conditions serves as a mathematical description of the problem. Long-wave expansions are carried out and an evolution equation of the Kuramoto-Sivashinsky type governing propagation of weak surface perturbations is derived. The critical values of the Reynolds number are determined explicitly and linear stability is investigated. It is shown that the electrical field provides a destabilizing effect on the film flow while the magnetic field stabilizes it. The strongest stabilizing effect of the magnetic field in the presence of the electrical one can be achieved if it is purely longitudinal. The Karman-Polhausen integral boundary-layer theory is considered for the MHD-approach and main results are discussed. © Elsevier, Paris.

1. Introduction

In the steady laminar flow of a fluid layer of thickness d down an inclined plane, the gravitational acceleration is balanced by the effect of the viscosity of the fluid. The velocity is then parallel to the plane and has a parabolic profile with the vertex at the free surface. As a parameter characterizing the steady flow, one usually takes the Reynolds number $Re = \lambda c_0 / \nu_0$, where λ is the reference length down the plane, ν_0 is the viscosity of the fluid, $c_0 = g d^2 \sin \varphi / 3 \nu_0$ is the Nusselt velocity for the average velocity of a flat film, and φ is the angle of inclination of the plane. When artificial perturbing pulses are imparted to this layer, different types of waves are observed (Kapitza and Kapitza, 1949). Immediately beyond an inception region they are short near-sinusoidal capillary waves while further downstream long near-solitary waves with large tear-drop humps preceded by short, front-running waves appear. It was found that the onset of these waves is connected with long surface-wave instability, i.e. beyond a certain critical value Re_{crit} of the Reynolds number, perturbations of the flow are linearly unstable and lead to formation of waves (Nakaya, 1975). Linear stability theory gives a reasonably accurate prediction of the critical conditions for the onset of waves, although different values of Re_{crit} appear. The low-Reynolds number theory of Benjamin (1957) and Yih (1963) yields the leading order $Re_{crit} = 5 \cot \varphi / 6$ while the critical value within the high Reynolds number limit gives $Re_{crit} = \cot \varphi$ (Prokopiou, Cheng and Chang, 1991).

The impact of nonlinearity on the evolution of surface waves can be studied on the basis of a perturbation technique which gives a single equation for the surface height. A general wave dynamic equation describing weakly nonlinear surface waves on a viscous fluid moving down an inclined plane was derived by Topper and Kawahara (1978), taking into account both dispersive and dissipative properties of the fluid as well as the essential surface tension. This equation can be referred as the Kuramoto-Sivashinsky equation with dispersion

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or simply the Kawahara equation. However the most commonly used evolution equation which can provide a starting point for theoretical studies of film flows is the Kuramoto-Sivashinsky (KS) equation (Sivashinsky and Michelson, 1980). It does not include a dispersive term, however it was shown by Chang, Demekhin and Kopelevich (1993a) that more general equations suitable for nonlinear waves on viscous films have a behaviour similar to the KS equation.

Electrically conducting films flowing under action of an applied magnetic field have been studied since the middle 60s. Linear stability analysis for a transverse external magnetic field, negligible surface tension and almost horizontal films was carried out by Hsieh (1965) and Ladikov (1966). It was found that MHD-coupling increases the stability of the film and can serve as a powerful remedy in suppressing instabilities of the flow. The influence of a transverse magnetic field on the MHD gravity-capillary waves travelling down the surface of a falling conducting liquid film was investigated by Lu and Sarma (1967). Wave flows of an electrically conducting liquid film over a horizontal plane in tangential magnetic and electrical fields was studied by Gordeev and Murzenko (1990). They found that at a certain magnitude of an external electrical field (it must be strong enough) the flow suffers from instability without any connection to the Reynolds number.

Here we consider a more complete extension of the analysis developed for the ordinary fluid flows to the MHD approach than has been done before. This means that we study the linear stability and propagation of long nonlinear waves on the surface of a thin layer of an electrically conducting fluid flowing down an isolated inclined plane under the action of constant crossed electrical and magnetic fields, while the magnetic field has both tangential and longitudinal components. The problem we will address is whether the simultaneous effect of electrical and magnetic fields may essentially affect the critical conditions for the onset of long-wave instability and features of nonlinear surface waves over a flow down an inclined plane.

We have also several practical reasons to consider this problem. Film flows of conducting fluids serve in elements of nuclear energy equipment and other technological applications, as in metallurgy (Glukhikh, Tananaev and Kirilov, 1987). Application of thin conducting and magnetic films in different cooling systems is also very promising (Blum, Mayorov and Tsebers, 1989). The generation and development of surface waves can drastically affect the properties of such systems. In some applications it is desirable to promote surface waves; and in others to suppress them. It is worth noting that the effect of external body forces can accelerate or impede the onset of waves over a flow, as is observed in convection (Benguria and Depassier, 1987). It is well known that the effect of the Lorentz force causes a loss of energy in the MHD-flow, and anisotropy of propagation of the waves due to the interaction of the flow and the external magnetic or electrical field (Shercliff, 1969, Debnath and Basu, 1978, Hofman, 1983). Therefore we suggest that the Lorentz force may essentially affect the critical conditions for the onset of long-wave instability in conducting films.

2. Formulation of the problem

We consider a finite layer of an electrically conducting liquid flowing down an inclined plane under the action of constant electrical and magnetic fields (the geometry of the flow is depicted in Figure 1). The bottom of the channel is supposed to be nonconducting, the upper half space is filled with air with negligible density and conductivity. An unperturbed electrical field $\mathbf{E}_0 = (0, E_0, 0)$ and magnetic field $\mathbf{H}_0 = (H_0 \cos \theta, 0, H_0 \sin \theta)$, where θ is the angle of inclination of the magnetic field, exists everywhere, a constant gravitational field acts upon the layer. It is assumed that the magnetic Reynolds number $R_m = \mu \sigma v_0 \lambda \ll 1$ (where μ is the magnetic permeability and σ is the conductivity of the fluid) and then the low magnetic Reynolds number limit for the MHD equations is used in the analysis (Branover and Tsinober, 1970; Korsunsky, 1997a). The set of equations that describes the motion of the fluid is:

$$\begin{aligned}
 (1) \quad \rho_0(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) &= -\nabla p + \rho_0 v_0 \Delta \mathbf{v} + \mu \sigma (\mathbf{E} + \mu \mathbf{v} \times \mathbf{H}_0) \times \mathbf{H}_0 + \rho_0 \mathbf{g}, \\
 \nabla \cdot \mathbf{v} &= 0, \quad \mathbf{E} = \mathbf{E}_0 - \nabla \Phi, \\
 \Delta \Phi &= \mathbf{H}_0 \cdot (\nabla \times \mathbf{v}),
 \end{aligned}$$

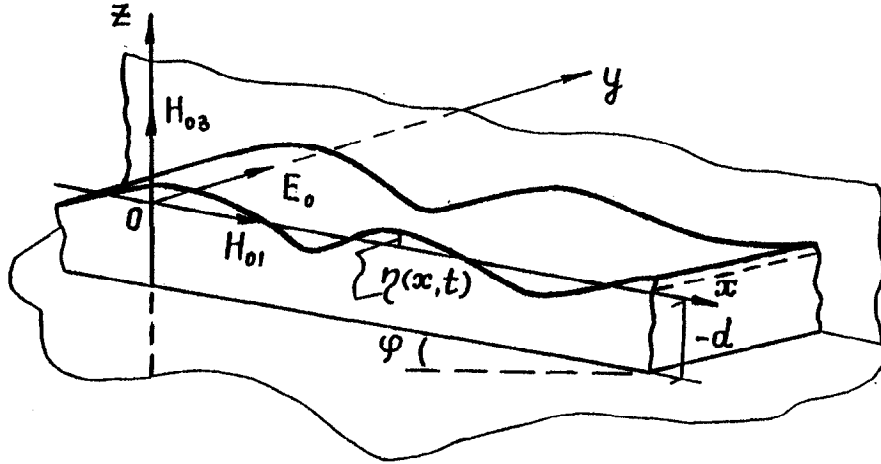


Fig. 1. – Geometry of the problem.

where $\mathbf{g} = (g \sin \varphi, 0, -g \cos \varphi)$, Φ is the potential of the electrical field. We restrict consideration to the two-dimensional case: $\mathbf{v} = (u, 0, w)$, $\partial/\partial y = 0$ and then $\Phi = \text{const}$, $\mathbf{E} = \mathbf{E}_0$. The boundary conditions are kinematic and dynamic conditions on the upper surface

$$\begin{aligned}
 (2) \quad z = \eta(x, t) : w &= \eta_t + u\eta_x, \quad (u_z + w_x)(1 - \eta_x^2) + 2\eta_x(w_z - u_x) = 0, \\
 p - p_a &= 2\rho_0 v_0 (1 + \eta_x^2)^{-1} (w_z - \eta_x(u_z + w_x) + \eta_x^2 u_x) - \Sigma_0 \eta_{xx} (1 + \eta_x^2)^{-3/2}
 \end{aligned}$$

and impermeability conditions on the bottom

$$(3) \quad z = -d : \quad u = 0, \quad w = 0.$$

In (2) Σ_0 is the surface tension, ρ_0 is the density of the fluid.

Let us introduce nondimensional quantities taking into account the intrinsic difference in scales for longitudinal and tangential coordinates and components of the velocity, as follows:

$$(4) \quad u^* = \frac{u}{c_0}, \quad x^* = \frac{x}{\lambda}, \quad z^* = \frac{z}{d}, \quad w^* = \frac{w\lambda}{dc_0}, \quad p^* = \frac{p - p_a}{\rho_0 c_0^2}, \quad t^* = \frac{tc_0}{\lambda}, \quad \eta^* = \frac{\eta}{d}$$

where $c_0 = gd^2 \sin \varphi / 3v_0$ is the Nusselt velocity, p_a is the atmospheric pressure. Then we rewrite system (1)-(3) in dimensionless form (superscripts are skipped henceforth)

$$\begin{aligned}
 (5) \quad u_t + uu_x + wu_z &= -p_x + \frac{1}{\text{Re}} u_{xx} + \frac{1}{\beta \text{Re}} u_{zz} + R + \frac{\alpha Ha^2 \sin \theta}{\beta \text{Re}} \\
 &\quad + \frac{Ha^2 \sin 2\theta}{2\sqrt{\beta} \text{Re}} w - \frac{Ha^2 \sin^2 \theta}{\beta \text{Re}} u, \\
 w_t + uw_x + ww_z &= -\frac{1}{\beta} p_z + \frac{1}{\text{Re}} w_{xx} + \frac{1}{\beta \text{Re}} w_{zz} + \frac{Ha^2 \sin 2\theta}{2\beta \sqrt{\beta} \text{Re}} u \\
 &\quad - \frac{Ha^2 \cos^2 \theta}{\beta \text{Re}} w - \frac{Fr}{\beta} - \frac{\alpha Ha^2 \cos \theta}{\beta \sqrt{\beta} \text{Re}}, \\
 u_x + w_z &= 0
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad z = -1 : \quad u &= w = 0, \\
 z = \eta(x, t) : \quad w &= \eta_t + u\eta_x, \quad u_z + \beta w_x = \frac{4\beta u_x \eta_x}{1 - \beta \eta_x^2}, \\
 p &= \frac{2}{\text{Re}(1 + \beta \eta_x^2)} (w_z - \eta_x(u_z + \beta w_x) + \beta u_x \eta_x^2) - \frac{We \beta \eta_{xx}}{(1 + \beta \eta_x^2)^{3/2}},
 \end{aligned}$$

where $\beta = \frac{d^2}{\lambda^2}$, $\alpha = \frac{E_0}{c_0 \mu H_0}$, $\text{Re} = \frac{\lambda c_0}{v_0}$ is the Reynolds number, $Ha = \mu H_0 d \sqrt{\frac{\sigma}{\rho_0 v_0}}$ is the Hartman number, $We = \frac{\Sigma_0}{\rho_0 c_0^2 d}$ is the Weber number, $Fr = \frac{gd \cos \varphi}{c_0^2}$ is the Froude number and, as it is easy to see, $Fr = \frac{3 \cot \varphi}{\text{Re} \sqrt{\beta}}$, $R = \frac{g \lambda \sin \varphi}{c_0^2} = \frac{3}{\beta \text{Re}}$. Here parameter α describes the influence of the electrical field, Ha serves as a measure of relative strength of the magnetic field and viscosity, We describes the influence of the surface tension, β is the parameter specifying dispersion.

To complete our formulation, we introduce in (5), (6) a redefined pressure P and Froude number F as follows

$$(7) \quad P = p + Fz, \quad F = Fr + \frac{\alpha Ha^2 \cos \theta}{\sqrt{\beta} \text{Re}}$$

absorbing two last terms in second equation of (5). Now the last boundary condition in (6) takes the form

$$(8) \quad P = F\eta + \frac{2}{\text{Re}(1 + \beta \eta_x^2)} (w_z - \eta_x(u_z + \beta w_x) + \beta u_x \eta_x^2) - \frac{We \beta \eta_{xx}}{(1 + \beta \eta_x^2)^{3/2}}.$$

3. Shallow water theory and linear stability

We consider propagation of long wave perturbations assuming that $\beta \ll 1$ (shallow water theory). Since we have this small parameter one or another perturbation technique can be applied for the study. However we should first determine the basic flow in the absence of any perturbations.

3.1. BASIC FLOW

The basic-flow solution to the system (5), (6), (8) assumes that $u = U_s(z)$, $P = P_s(z)$, $w = 0$, $\eta = 0$, where $U_s(z)$ and $P_s(z)$ are solutions to the problem

$$\begin{aligned}
 (9) \quad & U_s'' - Ha^2 \sin^2 \theta U_s + 3 + \alpha Ha^2 \sin \theta = 0, \\
 & P_s' = \frac{Ha^2 \sin 2\theta}{2\sqrt{\beta Re}} U_s \\
 & U_s(-1) = 0, \quad U_s'(0) = 0, \quad P_s(0) = 0
 \end{aligned}$$

which are

$$\begin{aligned}
 (10) \quad & U_s = D \left(1 - \frac{\cosh(zHa \sin \theta)}{\cosh(Ha \sin \theta)} \right), \quad D = \frac{\alpha}{\sin \theta} + \frac{3}{Ha^2 \sin^2 \theta} \\
 & P_s = \frac{DHa \cos \theta}{\sqrt{\beta Re}} \left(zHa \sin \theta - \frac{\sinh(zHa \sin \theta)}{\cosh(Ha \sin \theta)} \right)
 \end{aligned}$$

We note that average velocity of the basic flow $U_{aver} = \int_{-1}^0 U_s dz$ is

$$(11) \quad U_{aver} = D \left(1 - \frac{\tanh(Ha \sin \theta)}{Ha \sin \theta} \right)$$

if $\theta \neq 0$ and $U_{aver} = 1$ if $\theta = 0$. Plots of the function (11) with respect to $Ha \sin \theta$ for different values of $\hat{\alpha} = \alpha \sin^{-1} \theta$ are depicted in Figure 2. We see that in the absence of or for a weak electrical field an applied magnetic field provides a reduction in the average velocity. This tendency is opposite when the electrical field is quite strong. This is due to the fact that the electrical field provides the Lorentz force which accelerates the flow downstream while the part of Lorentz force due to interaction of the velocity and magnetic field brakes it.

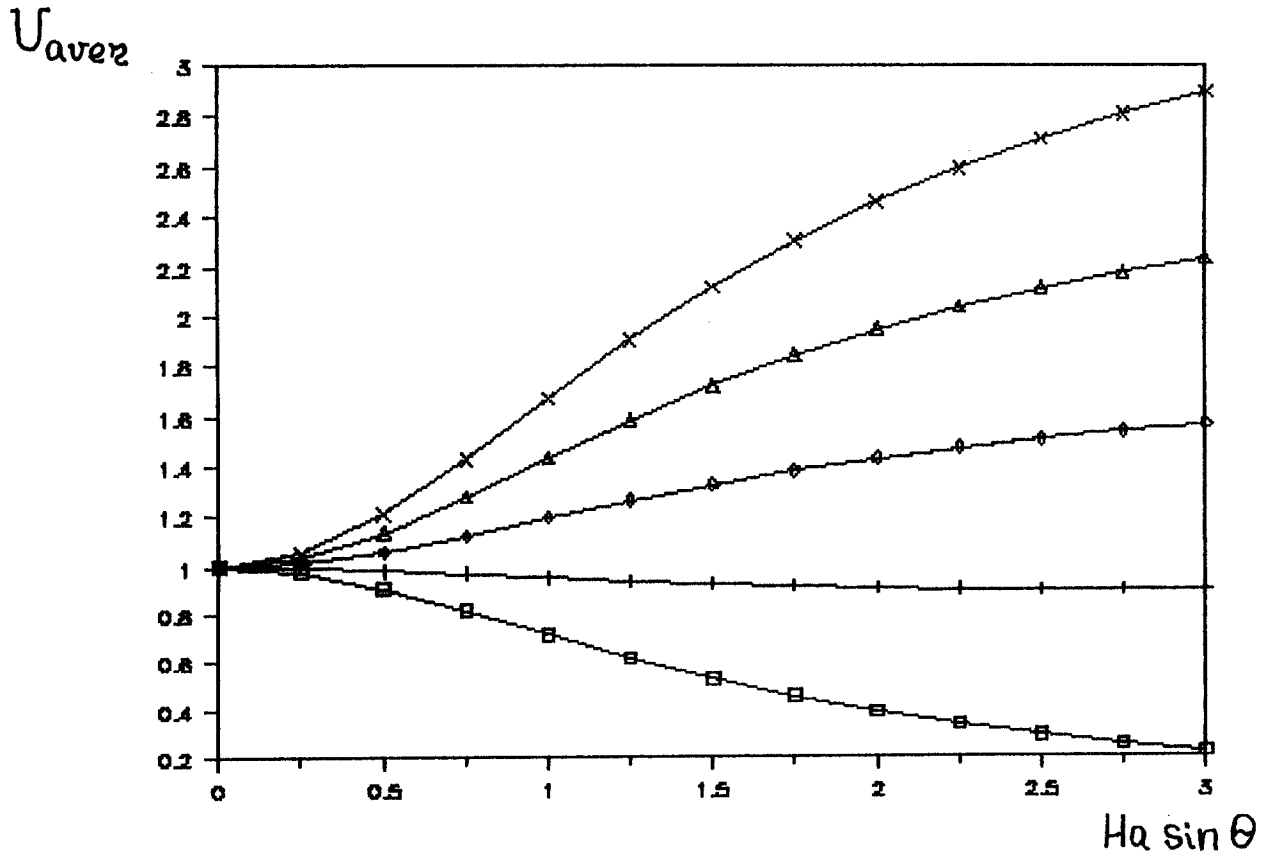


Fig. 2. – Averaged velocity of the basic flow versus $Ha \sin \theta$ at different values of $\hat{\alpha} = \alpha \sin^{-1} \theta$: (\square) – $\hat{\alpha} = 0$; (+) 1; (\diamond) 2; (Δ) 3; (\times) 4.

3.2. PERTURBATION ANALYSIS

Now we impose a small perturbation of order $\varepsilon \ll 1$ on the free surface of the film: $\eta = \varepsilon \hat{\eta}$, which produces weak perturbations of the main flow:

$$(12) \quad u = U_s(z) + \varepsilon \hat{u}, \quad P = P_s(z) + \varepsilon \hat{p}, \quad w = \varepsilon \hat{w}$$

Here $\varepsilon = d/\lambda = \beta^{1/2}$. Then we expand the boundary conditions (6), (8) on the free surface near equilibrium and retain all terms up to order $O(\varepsilon^2)$. This yields

$$(13) \quad \begin{aligned} z = 0: \quad & w + \varepsilon \eta w_z = \varepsilon \eta_\tau + \varepsilon(u \eta_x + \varepsilon \eta \eta_x u_z), \\ & u_z + \varepsilon \eta u_{zz} + \beta w_x + \varepsilon \beta \eta w_{xz} = 4\beta \varepsilon \eta_x \frac{u_x + \varepsilon \eta u_{xz}}{1 - \beta \varepsilon^2 \eta_x^2}, \\ & P + \varepsilon \eta P_z = F \varepsilon \eta - We \beta \frac{\varepsilon \eta_{xx}}{(1 + \beta \varepsilon^2 \eta_x^2)^{-3/2}} + \frac{2}{\text{Re}(1 + \beta \varepsilon^2 \eta_x^2)} \\ & \cdot (w_z + \varepsilon \eta w_{zz} + \beta \varepsilon^2 \eta_x^2 u_x - \varepsilon \eta_x (u_z + \varepsilon \eta u_{zz} + \beta w_x + \beta \varepsilon \eta w_{xz})). \end{aligned}$$

Below we assign the following orders to the nondimensional parameters:

$$(14) \quad \beta = \varepsilon^2, \quad \text{Re} = O(\varepsilon^{-1}), \quad We = O(\varepsilon^{-2}), \quad Ha = O(1), \quad \alpha = O(1)$$

since we deal with the long-wave approximation at moderate Reynolds number, low Hartman number and with relatively essential surface tension. The estimates (14) give that the Froude number $F = O(1)$ and

$$(15) \quad F = \frac{3 \cot \varphi}{\text{Re}} + \frac{\alpha Ha^2 \cos \theta}{\text{Re}}$$

Introducing the relations (10), (12), (14) and the new scaled variables in the moving frame $\tau = \varepsilon t$, $\xi = x - At$ into (5), (6) and (13), we obtain the following set of equations and boundary conditions with respect to the perturbed values (the caret will be dropped henceforth):

$$(16) \quad \begin{aligned} & \varepsilon u_\tau + (U_s - A)u_\xi + \varepsilon u u_\xi + U_s' w + \varepsilon w u_z \\ & = -p_\xi + \frac{\varepsilon}{\text{Re}} u_{\xi\xi} + \frac{1}{\varepsilon \text{Re}} u_{zz} + \frac{Ha^2 \sin 2\theta}{2\text{Re}} w - \frac{Ha^2 \sin^2 \theta}{\varepsilon \text{Re}} u, \\ & \varepsilon w_\tau + (U_s - A)w_\xi + \varepsilon u w_\xi + \varepsilon w w_z \\ & = -\frac{1}{\varepsilon^2} p_z + \frac{\varepsilon}{\text{Re}} w_{\xi\xi} + \frac{1}{\varepsilon \text{Re}} w_{zz} + \frac{Ha^2 \sin 2\theta}{2\varepsilon^2 \text{Re}} u - \frac{Ha^2 \cos^2 \theta}{\varepsilon \text{Re}} w, \\ & u_\xi + w_z = 0 \end{aligned}$$

$$(17) \quad \begin{aligned} z = 0: \quad & w + \varepsilon \eta w_z = \varepsilon \eta_\tau + \varepsilon u \eta_\xi + (U_s - A)\eta_\xi, \\ & u_z + \varepsilon \eta u_{zz} + \varepsilon^2 w_\xi + \eta U_s'' = 0, \\ & p + \varepsilon \eta p_z = (F - P_s')\eta - T \eta_{\xi\xi} + \frac{2\varepsilon}{\text{Re}} w_z, \\ z = -1: \quad & u = w = 0 \end{aligned}$$

In (17) we have neglected small terms of order $O(\varepsilon^2)$ and higher. Seeking solutions to the problem (16), (17) in form of the asymptotic expansions

$$(18) \quad f = f_0 + \varepsilon f_1 + \dots$$

we get to order $O(1)$

$$\begin{aligned} u_{0zz} - N^2 u_0 &= 0, \quad p_{0z} = \text{Re}^{-1} N^2 \cot \theta u_0, \quad u_{0\xi} + w_{0z} = 0 \\ z = -1 : \quad u_0 &= w_0 = 0, \\ z = 0 : \quad w_0 &= (U_s - A)\eta_{0\xi}, \quad u_{0z} = -\eta_0 U_s'', \quad p_0 = F^* \eta_0 - T\eta_{0\xi\xi} \end{aligned}$$

This yields

$$(19) \quad \begin{aligned} u_0 &= \frac{DN\eta_0}{\cosh^2(N)} \sinh(N(z+1)), \quad w_0 = \frac{D\eta_{0\xi}}{\cosh^2(N)} (1 - \cosh(N(z+1))), \\ p_0 &= F^* \eta_0 - T\eta_{0\xi\xi} + \frac{\eta_0 DN^2 \cot \theta}{\text{Re} \cosh(N)} \left(\frac{\cosh(N(z+1))}{\cosh(N)} - 1 \right), \quad A = D \tanh^2(N) \end{aligned}$$

where we put for simplicity $N = Ha \sin \theta$ and $F^* = F - \frac{DN^2 \cot \theta}{\text{Re}} \left(1 - \frac{1}{\cosh(N)} \right)$.

To order $O(\varepsilon)$ we find, after rather formidable calculations (details are provided in the Appendix), functions $u_1(\tau, \xi, z)$ and $w_1(\tau, \xi, z)$ and, as a solubility condition, the following evolution equation for the film height from the unperturbed level

$$(20) \quad \eta_{0\tau} + \gamma_1 \eta_0 \eta_{0\xi} + \gamma_2 \eta_{0\xi\xi} + \gamma_4 \eta_{0\xi\xi\xi} = 0$$

where the coefficients are

$$(21) \quad \gamma_1 = \frac{2DN \tanh(N)}{\cosh^2(N)}, \quad \gamma_4 = \frac{T \text{Re}}{N^2} \left(1 - \frac{\tanh(N)}{N} \right),$$

$$(22) \quad \begin{aligned} \gamma_2 &= \frac{\text{Re}}{N^2} \left(\frac{3D^2 N \tanh(N)}{2 \cosh^2(N)} \left(1 - \frac{\tanh(N)}{N} - \frac{\tanh^2(N)}{3} \right) \right. \\ &\quad \left. - \left(1 - \frac{\tanh(N)}{N} \right) F \right) \\ &\quad + \frac{D \cot \theta}{N \cosh^2(N)} (N(\cosh^2(N) + 2) - \sinh(N)(2 + \cosh(N))) \end{aligned}$$

This is the well-known Kuramoto-Sivashinsky equation obtained, particularly, for the film flows of viscous fluid (Sivashinsky and Michelson, 1980). We will discuss it later.

3.3. LINEAR STABILITY

Linear periodic perturbations

$$(23) \quad \eta_0 = \text{const} + \chi \exp(i(k\xi - \omega\tau)), \quad \chi \ll 1$$

are governed by equation (20) only if

$$(24) \quad \omega = \text{const} \cdot k + ik^2(\gamma_2 - \gamma_4 k^2).$$

This yields that the system considered exhibits long-wave instability for $k^2 < \gamma_2/\gamma_4$ while short waves ($k^2 > \gamma_2/\gamma_4$) are stable. The critical wave number is $k_{\text{crit}}^2 = \gamma_2/\gamma_4$ and the wave number of the fluctuations with maximum growth rate is $k_m^2 = \gamma_2/2\gamma_4$. The critical condition for the onset of waves then is $\gamma_2 = 0$ or $\text{Re} = \text{Re}_{\text{crit}}$, where

$$(25) \quad \text{Re}_{\text{crit}} = \left(3\cot\theta + \frac{\alpha N^2 \cot\theta}{\sin\theta} \right) \frac{2\cosh^2(N)}{D^2 \tanh(N)} \frac{1 - N^{-1} \tanh(N)}{(3N - 3\tanh(N) - N \tanh^2(N))} \\ + \frac{2N \cot\theta}{D \tanh(N)} \cdot \frac{N(\cosh^2(N) + 2) - \sinh(N)(2 + \cosh(N))}{(3N - 3\tanh(N) - N \tanh^2(N))}$$

with zero critical wave number. Here

$$D = \frac{3}{N^2} + \frac{\alpha}{\sin\theta}.$$

To verify the mathematics we put $\theta = \pi/2$ in (25) (the case of a tangential magnetic field) and obtain

$$(26) \quad \text{Re}_{\text{crit}} = \cot\varphi \frac{2\cosh^2(Ha)}{D^2 Ha \tanh(Ha)} \left(1 - \frac{\tanh(Ha)}{Ha} \right) \left(1 - \frac{\tanh(Ha)}{Ha} - \frac{\tanh^2(Ha)}{3} \right)^{-1}$$

At $\alpha = 0$ this relationship is in full correspondence with the results of Ladikov (1966) and Hsieh (1965). The limiting tendances $\alpha \rightarrow 0$, $Ha \rightarrow 0$ in (26) gives

$$\text{Re}_{\text{crit}} = \frac{5}{6} \cot\varphi$$

as was found by Topper and Kawahara (1978), Yih (1963) and Johnson (1972). The results of Gordeev and Murzenko can be obtained from (26) by putting $\varphi = 0$ (in this case one should remember that we calculate Reynolds number using Nusselt velocity as a characteristic velocity in (4)). Finally we note that in the absence of electrical and magnetic fields the Kuramoto-Sivashinsky equation (20) reduces exactly to the equation obtained by Nepomnyaschy (1974) and Lin (1974) for ordinary films.

For a longitudinal magnetic field ($\theta = 0$) we get from (25) after taking the appropriate limiting tendency

$$(27) \quad \text{Re}_{\text{crit}} = \frac{5}{6} \left(\cot\varphi + \frac{\alpha Ha^2}{3} \right)$$

Plots of the critical Reynolds number (26) with respect to Hartman number at different values of α are represented in Figure 3a. We see that in a relatively weak electrical field application of a magnetic field gives an essential rise of the critical Reynolds number. Increasing α makes this tendency smoother and, moreover, within a certain range of Ha the critical Reynolds number decreases. Thus we conclude that electrical field gives a destabilizing impact on the wave flow while magnetic field exhibits a strong stabilizing effect.

Information about the influence of the direction of application of the magnetic field on the stability can be gained from the following consideration. Since the critical Reynolds number (25) is a monotonously increasing function with respect to N , we provide an asymptotic expansion for (25) at $N = Ha \sin\theta \ll 1$ and obtain

$$\text{Re}_{\text{crit}} \approx \frac{5}{6} \cdot \frac{\cot\varphi + \frac{\alpha Ha^2 \cos\theta}{3}}{(1 + \frac{\alpha Ha^2 \sin\theta}{3})^2} + \frac{11}{24} \cdot \frac{Ha^2 \sin\theta \cos\theta}{1 + \frac{\alpha Ha^2 \sin\theta}{3}}.$$

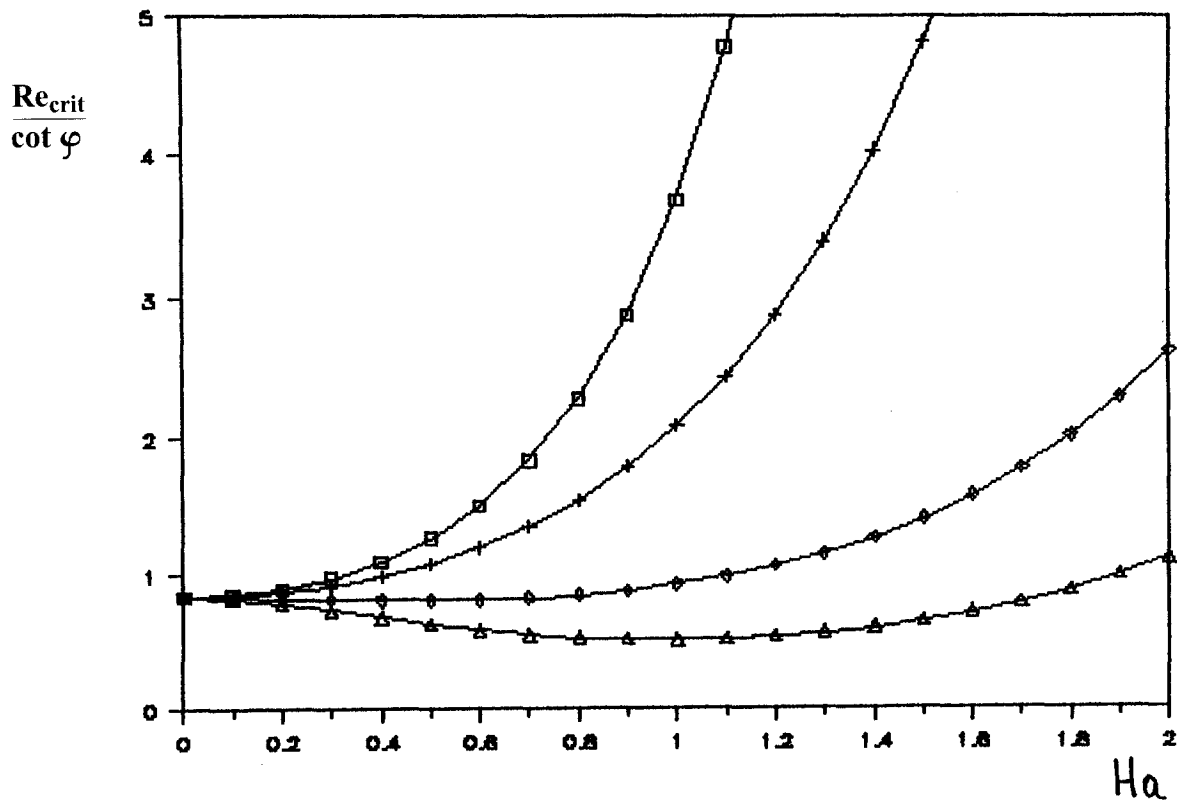


Fig. 3a. – Critical Reynolds number versus Hartman number at different values of parameter α : (\square) – $\alpha = 0$; (+) 1; (\diamond) 3; (Δ) 5.

Now it is easy to see that in the presence of electrical field ($\alpha \neq 0$) maximum growth of the critical Reynolds can be achieved at $\theta = 0$ (longitudinal magnetic field). The critical Reynolds number in this case is given explicitly by (27). In the absence of an electrical field ($\alpha = 0$) the same result corresponds to the case $\theta = \pi/4$.

Plots on Figure 3b represent critical Reynolds number (25) with respect to $N = Ha \sin \theta$ at fixed $\hat{\alpha} = \alpha \sin^{-1} \theta = 1$ and for different values of the parameter $\Omega = \cot \theta / \cot \varphi$. We take φ fixed and $\varphi \neq 0, \pi/2$ while θ changes from $\pi/2$ ($\Omega = 0$) till 0 ($\Omega \rightarrow \infty$). We found that the critical Reynolds number increases with growth of Ω . This means that the strongest stabilizing impact of the magnetic field can be achieved in the longitudinal case as was mentioned above.

Physical explanation of these results follows from the fairly simple geometrical consideration. The basic flow has only a downstream component. For the perturbation, this velocity component is of order of magnitude larger than transverse velocity. In the case of transverse magnetic field, the part of the Lorentz force connected with the electrical field accelerates the flow downstream while the part of the Lorentz force connected with the interaction of the velocity and magnetic fields is directed upstream and brakes it. The magnetic field lines act roughly like elastic strings which tend to resist the deviations in the normal velocity. These deviations become more and more difficult to manage as the field strength increases and the elastic rings stiffen. We conclude that an electrical field destabilizes the film flow whereas a magnetic field stabilizes it at not too large α . At large α and moderate Ha the effect of the electrical field prevails over the magnetic field. In the case of a longitudinal magnetic field the basic flow does not depend on applied electromagnetic fields since the Lorentz force has only a transverse component. This component is connected with the electrical field and acts like an additional pressure resisting development of perturbations.

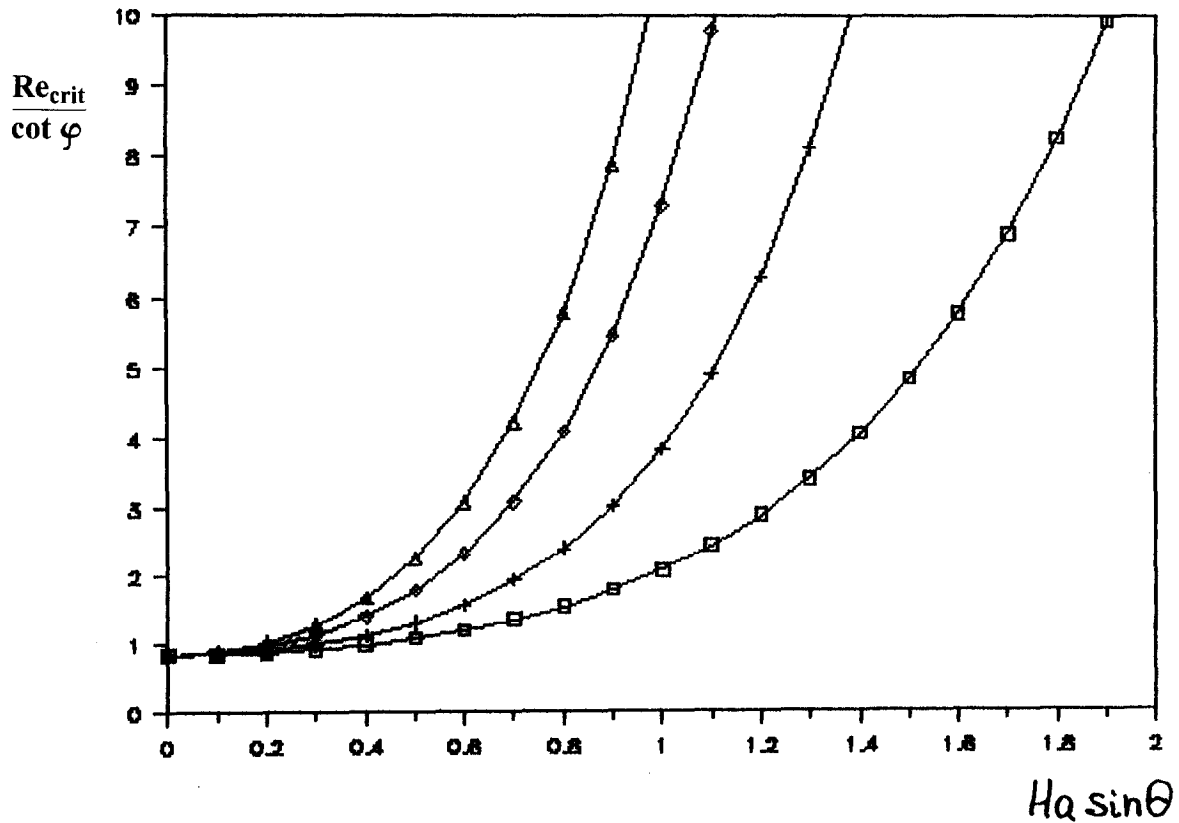


Fig. 3b. – Critical Reynolds number versus $Ha \sin \theta$ at $\hat{\alpha} = 1$ and for different values of $\Omega = \cot \theta / \cot \varphi$: (\square) – $\Omega = 0$; (+) 1; (\diamond) 3; (Δ) 5.

3.4. THE CASE OF NEGATIVE ALPHA

When the orientation of the electrical field is of opposite sign to that shown in Figure 1, we should change α to $-\alpha$ everywhere. This technical substitution raises qualitatively new results which intrinsically depend on the angle of inclination of the magnetic field.

First, starting with the basic-flow solution (10) at $\theta \neq 0$, it is possible now that $D \leq 0$ when $\alpha Ha^2 \sin \theta \geq 3$. This means that application of the Lorentz force can stop and even turn back the basic flow.

If $D = 0$ then equation (20) loses its sense as an evolution equation for weakly nonlinear waves and additional careful consideration should be provided. Similar to the problem of surface wave propagation in a two-layered fluid, where the nonlinear term disappears for equal depths of the layers, higher-order nonlinear terms will be involved, transforming the KdV-like equation into the modified KdV-like equation with cubic nonlinearity (Korsunsky, 1997a). Such analysis is beyond our scope here. However the linear stability analysis is still valid, although the critical condition does not now involve Reynolds number. One can easily show that for $D = 0$ ($\alpha Ha^2 \sin \theta = 3$ and $\theta \neq 0$) in (22), we have

$$(28) \quad \gamma_2 = -\frac{3}{N^2} \left(1 - \frac{\tanh(N)}{N} \right) (\cot \varphi - \cot \theta)$$

and the critical condition becomes

$$(29) \quad \varphi = \theta.$$

It is to be emphasized that from the physical point of view the equality $\alpha Ha^2 \sin \theta = 3$ can be hardly realized in practical applications since even for such highly conducting media as liquid metals, applied electrical and magnetic fields would have to be extremely strong to satisfy this condition. The most reasonable case when $\alpha Ha^2 \sin \theta = 3$ is possible, assumes $\theta = \pi/2$, $\varphi \rightarrow 0$, i.e. a tangential applied magnetic field and practically horizontal film.

If $\theta = 0$ (as in the case of a longitudinal magnetic field) the basic flow is simply parabolic

$$(30) \quad U_s = \frac{3}{2}(1 - z^2),$$

and the coefficients in equation (20) become

$$(31) \quad \gamma_1 = 6, \quad \gamma_2 = \frac{\text{Re}}{3} \left(\frac{18}{5} - F \right), \quad \gamma_4 = \frac{T\text{Re}}{3},$$

where, as follows from (15).

$$F = \frac{3\cot\varphi}{\text{Re}} + \frac{\alpha Ha^2}{\text{Re}}.$$

Then the modelling equation (20) is still valid even for negative alpha and the stability condition becomes

$$(32) \quad \text{Re}_{\text{crit}} = \frac{5}{6} \left(\cot\varphi - \frac{|\alpha|Ha^2}{3} \right).$$

The critical Reynolds number now decreases drastically with application of the electromagnetic field and is equal to zero when

$$(33) \quad 3\cot\varphi = |\alpha|Ha^2.$$

We conclude that for a given fluid submerged in a longitudinal magnetic field and at negative alpha, there is always a critical angle of inclination of the plane which leads to completely unstable flows.

4. Weakly nonlinear waves

4.1. KURAMOTO-SIVASHINSKY EQUATION

Let us introduce a re-scaled film height $H = \gamma_1 \eta_0$. Then equation (20) gives

$$(34) \quad H_\tau + HH_\xi + \gamma_2 H_{\xi\xi} + \gamma_4 H_{\xi\xi\xi} = 0$$

This Kuramoto-Sivashinsky (KS) equation has been studied in great depth (a sketch of the different solutions can be found in Chang, 1994) and the most detailed analysis of the KS equation has been provided by Chang, Demekhin and Kopelevich (1993a). All stationary wave families of the KS equation have now been constructed. There are actually an infinite number of such families, which end with a solitary wave with different number of humps. In addition one exact solution was found by Kudryashov (1988) using the Backlund transformation technique

$$(35) \quad H = -\frac{15}{19} \gamma_2 K(9f - 11f^3 - 2), \quad f = \tanh \left[\frac{K}{2} \left(\xi + \frac{30}{19} \gamma_2 K \tau \right) \right]$$

where $K^2 = 11\gamma_2/19\gamma_4$. This is a particular solitary wave solution with a local maximum and minimum connecting two constant states at $\xi \rightarrow \pm\infty$.

Within the framework of this paper it is to be noted here that the effect of MHD interaction does not modify the form of the modelling evolution equation for weakly nonlinear surface waves. This is in sharp contrast with the similar problem of surface wave propagation in a finite layer of ideal conducting fluid (Hofman, 1983; Korsunsky, 1997a). There an application of a magnetic field gives rise to additional dissipative terms in the basic KdV equation, transforming it into a perturbed Burgers-KdV equation which has qualitatively new solutions compared to the case when the magnetic field is turned off. In contrast, here the MHD-parameters Ha and α are shown up through the coefficients of the modelling KS equation and can only shift areas of existence of the particular solutions and, eventually, affect the parameters of the wave (velocity, height, etc.) as in (35), where γ_2, γ_4 are both now functions of α and Ha .

Qualitative information about the structure of the stationary solutions of the KS equation (34) can be obtained within the framework of dynamic systems theory. Since for the solutions of (34) of the form $H = H(\zeta)$, $\zeta = \xi - S\tau$ the stationary KS equation

$$(36) \quad \gamma_4 W''' + \gamma_2 W' - SW + W^2 = W_0,$$

where $H = 2W$, W_0 is an integration constant, can be transformed into a three-dimensional dynamical system $(x_1, x_2, x_3) = (W, W', W'')$, such that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, the stationary waves become closed trajectories (limit cycles) in the three-dimensional phase space. Two primary nonlinear wave families (G_1 and G_2) were found. A wave denoted G_1 has a smaller phase velocity than the linear wave of the same wave number and a smaller average thickness than the Nusselt value, while another, denoted G_2 travels faster and has a larger average film thickness. Solitary waves in the G_2 family are called positive because of their solitary humps, but the G_1 solitary waves are termed negative due to their solitary dips. The G_1 is unstable for long wavelengths and appears only at short wavelengths, while the G_2 family predominates for long waves. Additional standing-wave families bifurcate from these two in the parameter space of the dynamical system. Since the stationary KS equation (36) can be re-scaled to eliminate the coefficients γ_2, γ_4 , we conclude that the structure of nonlinear waves in the problem considered completely reduces to the analysis developed for the abstract KS equation with simple re-scaling of the map of the solutions represented by Chang, Demekhin and Kopelevich (1993a). Finally it is to be noted that the wave structure is in good agreement with the experimental data of Kapitza and Kapitza (1949) and Nakoryakov, Pokusaev and Shreiber (1990).

4.2. DISPERSION

An extended version of the KS equation (34) which includes a dispersion term and is very useful for studying solitary wave propagation and interaction in a film flow can be obtained for specific orders of nondimensional parameters in the following form (Chang, 1994; Liu and Gollub, 1994)

$$(37) \quad H_\tau + HH_\xi + \gamma_2 H_\xi + \gamma_3 H_{\xi\xi\xi} + \gamma_4 H_{\xi\xi\xi\xi} = 0,$$

where $H = \gamma_1 \eta_0$. In the problem under consideration this can be done, for example, if $\theta = \pi/2$, $Re = O(1)$ and for almost vertical or pure horizontal films (Korsunsky, 1997b; Gordeev and Murzenko, 1990) in a similar way to that used above for derivation of equation (20). In this case the coefficients of equation (37) take the form

$$\begin{aligned}
 \gamma_1 &= \frac{2DH \tanh(Ha)}{\cosh^2(Ha)}, \quad \gamma_4 = \frac{TRe}{Ha^2} \left(1 - \frac{\tanh(Ha)}{Ha} \right), \\
 \gamma_2 &= \frac{Re}{Ha^2} \left(\frac{3D^2 H \tanh(Ha)}{2\cosh^2(Ha)} \left(1 - \frac{\tanh(Ha)}{Ha} - \frac{\tanh^2(Ha)}{3} \right) \right. \\
 &\quad \left. - \left(1 - \frac{\tanh(Ha)}{Ha} \right) Fr \right), \\
 \gamma_3 &= \frac{D}{Ha^2 \cosh^3(Ha)} (\cosh(Ha) - 1)(2\cosh(Ha) - 2 + Ha \sinh(Ha)).
 \end{aligned}
 \tag{38}$$

Equation (37) bears the name of KS equation with dispersion or Kawahara equation. It has been investigated in detail (Kawahara and Toh, 1988; Chang, Demekhin and Kopelevich, 1993b). Particularly, it was shown that with relatively small dispersion this equation possesses smooth solitary wave solutions while strong dispersion gives an oscillatory tail in front of the solitary waves. Some of exact solutions to equation (37) were found by Kudryashov (1988, 1989). They can show a solitary wave with a single hump or kind-shaped waves related to the specific values of the coefficients of the equation. Namely, if $\Gamma = \gamma_3/\sqrt{\gamma_2\gamma_4}$ is equal to ± 4 it is a solitary wave, if $\Gamma = \pm 12/\sqrt{47}$, $\pm 16/\sqrt{73}$ it is a kink-shaped wave. Numerical simulation for the equation (37) at $\Gamma = 4$ shows that an initial noise-like perturbation transforms after some temporal interval into a wave system which consists of a number of solitary waves. Variation of Γ affects the shape of these waves. Particularly, increasing Γ gives increasing amplitudes for the solitons. However, there is a critical value of $\Gamma \approx 0,3$ below which a solitary wave structure does not exist. Probably at this value we can expect transition from the regular wave structure to stochastic systems and chaos.

5. Integral boundary layer theory

We now briefly discuss the deviation of the present long-wave theory at $Re = O(\varepsilon^{-1})$ from the integral formulation at $Re = O(\varepsilon^{-2})$. Let us consider the intermediate Reynolds number problems for the tangential magnetic field, so in (5), (6) we put $\theta = \pi/2$ and suggest $Re = O(\varepsilon^{-2})$. It was shown by Benjamin (1957) that in this model the z -momentum balance is assumed to be dominated by hydrostatic forces and, also, terms of order $O(\beta)$ can be omitted in the boundary conditions (here, as above, we take $\beta = \varepsilon^2$). The leading-order version of the equations of motion and the pertinent boundary conditions become

$$\begin{aligned}
 u_t + uu_x + wu_z &= -p_x + vu_{zz} + 3v + vHa^2(\alpha - u) \\
 p_z &= 0 \\
 u_x + w_z &= 0
 \end{aligned}
 \tag{39}$$

$$\begin{aligned}
 z = -1 : \quad u &= w = 0, \\
 z = \eta(x, t) : \quad w &= \eta_t + u\eta_x, \quad u_z = 0, \quad p = Fr\eta - T\eta_{xx}
 \end{aligned}
 \tag{40}$$

where $v = 1/\beta Re = O(1)$. For $Re = O(\varepsilon^{-2})$ the long wave expansions are no longer valid because the nonlinear inertial terms are of the same order as the dominant viscous term. Then the approximate integral boundary-layer theory should be applied to equations (39), (40). The key approximation is the assumption of a velocity profile to allow integration of the equations in the z direction.

The Karman-Polhausen integral boundary-layer theory involves the following manipulations (Batchelor, 1967). Integration (39)-(40) from $z = -1$ to the interface $z = \eta$, one obtains the mass balance relationship and the averaged x -momentum equation

$$(41) \quad \begin{aligned} \eta_t + q_x &= 0, \\ q_t + Q_x &= T(1 + \eta)\eta_{xx} - Fr(1 + \eta)\eta_x + v(3 + \alpha Ha^2)(1 + \eta) \\ &\quad - vHa^2q - vu_z|_{z=-1}, \end{aligned}$$

where $q = \int_{-1}^{\eta} u dz$ is the local flow rate and $Q = \int_{-1}^{\eta} u^2 dz$. A specific profile must now be imposed in this theory. It is appropriate to use self-similar profile which corresponds to unperturbed flow

$$(42) \quad u = \frac{q}{\delta_1(1 + \eta)} f(Z), \quad f(Z) = \left(\alpha + \frac{3}{Ha^2} \right) \left(1 - \frac{\cosh(HaZ)}{\cosh(Ha)} \right), \quad Z = \frac{z - \eta}{1 + \eta}.$$

Then

$$(43) \quad Q = \frac{\delta q^2}{1 + \eta}, \quad \delta = \frac{\delta_2}{\delta_1^2}, \quad \delta_1 = \int_{-1}^0 f(Z) dZ, \quad \delta_2 = \int_{-1}^0 f^2(Z) dZ$$

and we find

$$vu_z|_{z=-1} = \frac{vqG}{\delta_1(1 + \eta)^2}, \quad G = \left(\alpha + \frac{3}{Ha^2} \right) Ha \tanh(Ha).$$

System (41) now takes the form

$$(44) \quad \begin{aligned} \eta_t + q_x &= 0, \\ q_t + \delta \frac{\partial}{\partial x} \left(\frac{q^2}{1 + \eta} \right) &= T(1 + \eta)\eta_{xx} - Fr(1 + \eta)\eta_x + v(3 + \alpha Ha^2)(1 + \eta) \\ &\quad - vHa^2q - \frac{vGq}{\delta_1(1 + \eta)^2} \end{aligned}$$

An obvious solution to (44) is the Nusselt flat-film solution:

$$(45) \quad q \equiv q_0 = \delta_1 = (\alpha + 3Ha^{-2})(1 - Ha^{-1}\tanh(Ha)), \quad \eta = 0.$$

Linearizing (44) about this base state and taking the partial derivative with respect to t for the first equation and with respect to x for the second equation, one can eliminate the deviation for q and obtain a single equation for the deviation of film height η :

$$(46) \quad \begin{aligned} \eta_{tt} + 2\delta q_0 \eta_{xt} + \delta q_0^2 \eta_{xx} + T\eta_{xxx} - Fr\eta_{xxt} + v(3 + 2G + \alpha Ha^2)\eta_x \\ + v(Ha^2 + Gq_0^{-1})\eta_t = 0 \end{aligned}$$

Introducing the normal mode $\eta = \exp(-i(kx - \omega t))$, where k is the wave number and ω is the wave frequency, to (46) yields the complex equation

$$(47) \quad \omega^2 - 2\delta q_0 \omega k + (\delta q_0^2 - Fr)k^2 = Tk^4 + ivB(\omega - ck),$$

where $B = Ha^3(Ha - \tanh(Ha))^{-1}$, $c = q_0(1 + 2Ha^{-1}\tanh(Ha))$. We shall focus only on the temporal instability problem with k real and ω complex. A simple manipulation of (47) yields that the condition for onset is

$$(48) \quad \begin{aligned} \text{Re} = \text{Re}_{\text{crit}} &= \frac{3\cot\varphi}{q_0^2} \left(\left(1 + 2\frac{\tanh(Ha)}{Ha} \right)^2 - \delta \left(1 + 4\frac{\tanh(Ha)}{Ha} \right) \right)^{-1}, \\ \text{Re}\{\omega\}/k &= c, \quad k_{\text{crit}} = 0. \end{aligned}$$

It should be noted that leading terms in the expansions of Re_{crit} and c at small Hartman number give $\text{Re}_{\text{crit}} \rightarrow \cot\varphi$, $c \rightarrow 3$ as $Ha \rightarrow 0$ which is in complete correspondence with the results of Prokopiou, Cheng and Chang (1991). The critical Reynolds number in (48) is different from (26) since we now consider the higher Reynolds number approach to the problem. But numerical calculations of this critical Reynolds as a function with respect to Hartman number at different values of the parameter α , depicted on Figure 4, show the same tendency as for the critical Reynolds number in (26) - in a relatively weak electrical field application of a magnetic field gives an essential rise of the critical Reynolds number, but increasing α makes this tendency smoother and within the certain range of Ha the critical Reynolds number decreases.

In the neighbourhood of the critical Reynolds number an expansion of (47) shows that one root is always stable, while the more unstable one yields

$$(49) \quad \begin{aligned} \text{Re}\{\omega\} &= kc - 2(c - \delta q_0)\text{Re}k^3\Delta B^{-2} + O(k^5) \\ \text{Im}\{\omega\} &= -\Delta k^2 B^{-1} + (T\text{Re}B^{-1} + \text{Re}(\Delta^2 + 4\text{Re}\Delta(c - \delta q_0)^2)B^{-3})k^4 \end{aligned}$$

where $\Delta = (c^2 - 2\delta q_0 c + \delta q_0^2)(\text{Re} - \text{Re}_{\text{crit}})$. It follows from (47) that there are two neutral linear modes with wave number zero and

$$(50) \quad k = \left[\frac{\Delta B^2}{TB^2 + \Delta^2 + 4\Delta\text{Re}(c - \delta q_0)^2} \right]^{1/2}$$

Weakly nonlinear analysis of the evolution of surface waves in this approach can be carried out on the basis of equations (44) in a similar way as in Prokopiou, Cheng and Chang (1991). Consequently, we transform (44) into a moving frame by the transformation $x \rightarrow x - Vt$ and omitting the time dependence in this new coordinate. Integrating the transformed equations once and using the condition that the Nusselt flat-film base state must be a solution for all V , one obtains the following relationship between the local flow rate and the film height for the travelling waves:

$$(51) \quad q = q_0 + V\eta,$$

Substituting (51) into a transformed (44) and expanding to second order in the deviation film height η , one obtains

$$(52) \quad \eta''' = \lambda_1\eta + \lambda_2\eta' + \alpha_1\eta^2 + \alpha_2\eta\eta' + O(|\eta|^3)$$

where

$$\begin{aligned} \lambda_1 &= \frac{Ha^3}{T\text{Re}(Ha - \tanh(Ha))} (V - c), \quad \lambda_2 = \frac{1}{T} (Fr - V^2 + 2\delta Vq_0 - \delta q_0^2) \\ \alpha_1 &= \frac{2Ha^2}{T\text{Re}} \left(V - \left(\alpha + \frac{3}{Ha^2} \right) \left(1 + \frac{\tanh(Ha)}{2Ha} \right) \right), \quad \alpha_2 = \frac{1}{T} (3Fr - 2V^2(1 - \delta)) \end{aligned}$$

This equation can be transformed into a first-order differential system the solution of which can be discussed within the framework of dynamic-system theory (Pumir, Manneville and Pomeau, 1983). Such analysis is

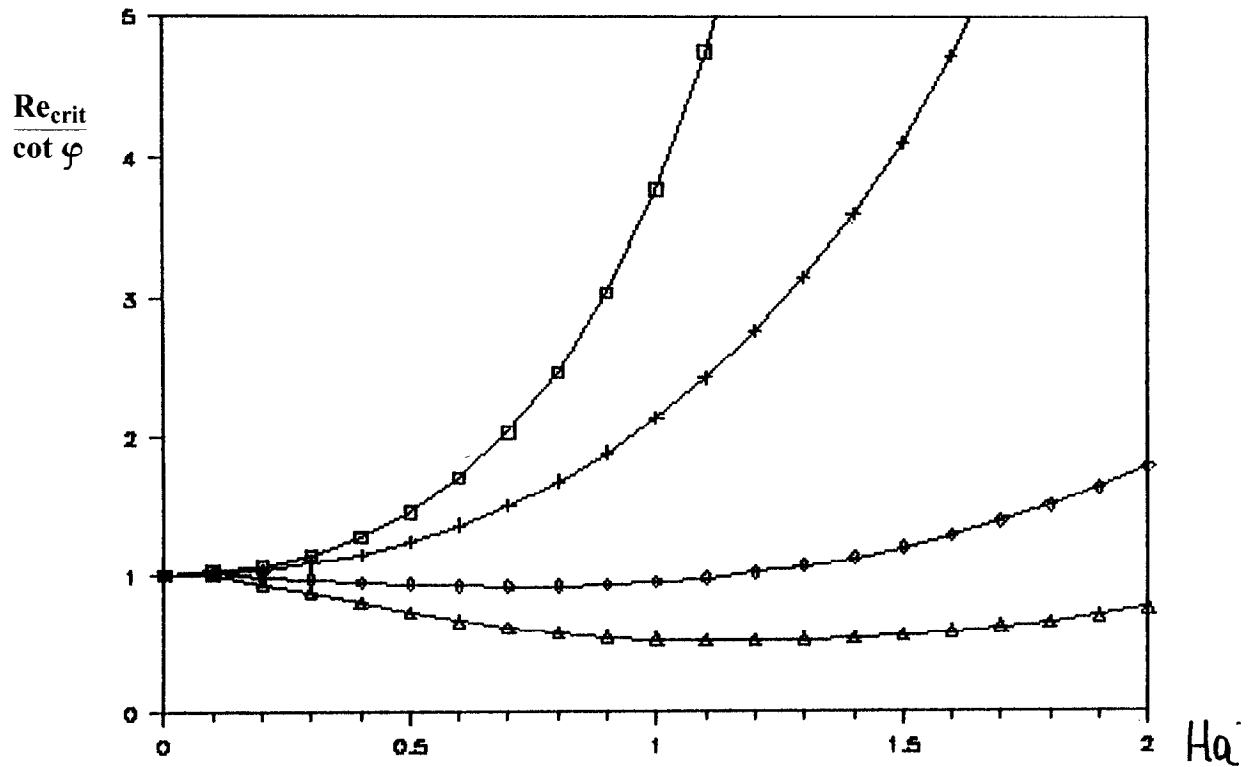


Fig. 4. – Critical Reynolds number versus Hartman number at different values of α and for transversal magnetic field within integral boundary-layer approach: (\square) – $\alpha = 0$; (+) 1; (\diamond) 3; (Δ) 5.

beyond our scope here. However it is worth noting that equation (52) has an exact solution which can be obtained in the following way.

Making use the substitutions

$$(53) \quad \alpha_2 \eta = U, \quad U' = \Psi(U)$$

we get an equation

$$(54) \quad \Psi^2 \Psi'' + \Psi \Psi'^2 = \lambda_1 U + \lambda_2 \Psi + \alpha^* U^2 + U \Psi$$

where $\alpha^* = \alpha_1 / \alpha_2$. The solution of this equation is

$$(55) \quad \Psi = a_1 U + a_2 U^{3/2}$$

if only $a_1 = \frac{4}{5} \alpha^*$, $a_2 = \frac{1}{3}$, $\lambda_1 = -\frac{48}{25} \alpha^{*3}$, $\lambda_2 = \frac{76}{25} \alpha^{*2}$. From (53) and (55) we find

$$(56) \quad \eta = \frac{\lambda_1}{\alpha_1 (1 + \text{const} \exp(-\frac{2}{5} \alpha^* (x - Vt)))^2}$$

This is a new kink-type solution connecting two basic states of the wave flow: $\eta = 0$ and $\eta = -\lambda_1 / \alpha_1$.

6. Summarizing remarks

Since the development of a wavy surface on a liquid flowing film sometimes is not desirable, it should be somehow suppressed. We have checked this possibility for the case when the liquid film is electrically conducting and affected by electrical and magnetic fields simultaneously. It was found that application of Lorentz force essentially affects linear stability of the flow. Namely, in a relatively weak electrical field application of a magnetic field gives noticeable rise of the critical Reynolds number. Increasing α makes this tendency smoother and, moreover, within the certain range of Ha the critical Reynolds number decreases. Thus we conclude that electrical field gives a destabilizing effect on the wave flow while a magnetic field exhibits a strong stabilizing effect. However this conclusion is valid only for the direction of the electrical field depicted on Figure 1. If it is directed in the opposite sense, the simultaneous action of electrical and magnetic fields destabilizes the flow.

Besides that, since weakly nonlinear waves in the MHD approach are governed by the classical KS equation or KS equation with dispersion, with modified coefficients due to electromagnetic parameters, we conclude that MHD-effects can only quantitatively affect the regimes of nonlinear wave propagation.

Finally, a brief discussion of an extension of integral boundary layer theory to the MHD approach, which was provided above, shows that the influence of the Lorentz force on long-wave instability at intermediate Reynolds number is qualitatively the same as in the limit of low Reynolds number.

Acknowledgements. – The author is grateful to the referees for their attention to the manuscript and useful remarks on the original paper.

APPENDIX A

Calculations for the first-order functions

Substitution of expansions (18) into (16), (17) gives to order $O(\varepsilon)$

$$\begin{aligned}
 (57) \quad u_{1zz} - N^2 u_1 &= -N^2 \cot \theta w_0 + \operatorname{Re}(p_{0\xi} + U'_s w_0 + (U_s - A) u_{0\xi}) \\
 \operatorname{Re} p_{1z} - N^2 \cot \theta u_1 &= w_{0zz} - N^2 \cot^2 \theta w_0 \\
 w_{1z} &= -u_{1\xi} \\
 z = -1 : \quad u_1 &= w_{1=0} \\
 z = 0 : \quad w_1 + \eta_0 w_{0z} &= \eta_0 \tau + (U_s - A) \eta_{1\xi} + u_0 \eta_{0\xi}.
 \end{aligned}$$

$$\begin{aligned}
 (58) \quad u_{1z} + \eta_1 U''_s + \eta_0 u_{0zz} &= 0 \\
 p_1 + \eta_0 p_{0z} &= (F - P'_s) \eta_1 - T \eta_{1\xi\xi} + \frac{2}{\operatorname{Re}} w_{0z}.
 \end{aligned}$$

As is usual, in order to get an evolution equation for such problems, it is enough to find solutions for the components of the velocity only. Then, using (10) and (19) we obtain

$$\begin{aligned}
 u_1 &= \eta_{0\xi} \operatorname{Re} \frac{D^2 \tanh(N)}{N \cosh^2(N)} - \frac{\operatorname{Re}}{N^2} (F^{**} \eta_{0\xi} - T \eta_{0\xi\xi\xi}) \\
 &\quad + \cosh(Nz)(z A_1(\xi) + B_1(\xi)) + \sinh(Nz)(z A_2(\xi) + B_2(\xi)) \\
 (59) \quad w_1 &= -z \left(\eta_{0\xi} \operatorname{Re} \frac{D^2 \tanh(N)}{N \cosh^2(N)} - \frac{\operatorname{Re}}{N^2} (F^{**} \eta_{0\xi} - T \eta_{0\xi\xi\xi}) \right) + M(\xi) \\
 &\quad - \frac{1}{N} (B_{1\xi} \sinh(Nz) + B_{2\xi} \cosh(Nz)) - \frac{A_{1\xi}}{N^2} (Nz \sinh(Nz) \\
 &\quad - \cosh(Nz)) - \frac{A_{2\xi}}{N^2} (Nz \cosh(Nz) - \sinh(Nz))
 \end{aligned}$$

where

$$F^{**} = F^* - \frac{DN^2 \cot \theta}{\operatorname{Re} \cosh^2(N)} (1 + \cosh(N))$$

and

$$A_1 = \frac{DN \cot \theta \tanh(N)}{\cosh(N)} \eta_{0\xi}, \quad A_2 = \left(\frac{\operatorname{Re} D^2 N \tanh(N)}{2 \cosh^3(N)} + \frac{DN \cot \theta}{\cosh(N)} \right) \eta_{0\xi}.$$

Now relationships (59) should be substituted into boundary conditions (58). This forms a homogenous system of four equations with respect to four unknown functions B_1 , B_2 , η_1 , M . The condition of existence of a nontrivial solution to this system gives the solubility condition - equation (20).

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(Received 3 April 1997;
revised 24 November 1997;
accepted 27 May 1998.)